8.2) Integration by U-substitution: part 2

- Sometimes when we substitute we end up with an "extra" x or two still left.
- In certain cases, these can be removed by an additional Substitution ...and we still end up with a manageable integral.

Example: $\int x\sqrt{x+1}\quad dx$

"extra"

$u = x+1$

$\frac{du}{dx} = 1$

$dx = du$

Solve for x

$x = u - 1$

$$\int \boxed{x} \sqrt{u}\, du$$

$$\int (u-1)\, u^{1/2}\, du$$

$$\int u^{\frac{3}{2}} - u^{1/2}\, du$$

$$\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

*more points = more work

↲ this!!!

↑closest to end of exam.

SUMMARY:

* Picking the 'correct' u **CAN** lead to a successful substitution.

* Picking u is an art, not a science. It takes practice.

* Not all problems can be solved by using u-substitution...what remains may still be too complicated to integrate.

$$\int_0^1 e^{-x^2}\, dx$$ ← Paper 2 GDC

Definite integrals with u-sub:

Example: Evaluate $\int_{X=\,0}^{X=\,4} \sqrt{2x+1} \quad dx$.

$v = 2x+1$

$\frac{dv}{dx} = 2 \rightarrow dx = \frac{dv}{2}$

Method 1:

1. Make u-sub and evaluate the indefinite integral first.
2. Substitute x's back in for u's.
3. Use the **evaluation theorem**. $F(b) - F(a)$

Example: (ctd).

$$\int \sqrt{v}\,\frac{dv}{2} = \frac{1}{2}\int v^{1/2}\,dv = \left[\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}\, v^{\frac{3}{2}}\right] = \left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]_0^4$$

limits come back

$$= \frac{1}{3}(9)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

! ! ! ! !

① $\sqrt[2]{9^{3}}$ ②

Method 2:

1. After picking $u = g(x)$ compute new "v-limits"
2. If x-limits are $x = a$ and $x = b \Rightarrow u$-limits: $v_1 = g(a)$ $v_2 = g(b)$
3. Use u-sub to compute integral then use **evaluation theorem**. ← never go back to x here

Example: (ctd).

$v = 2x+1$

New limits:

$x = 0 \rightarrow v = 2(0)+1 = 1$

$x = 4 \rightarrow v = 2(4)+1 = 9$

$$\frac{1}{2}\int_1^9 v^{\frac{1}{2}}\,dv$$

$$= \left[\frac{1}{3} v^{\frac{3}{2}}\right]_1^9 = \left(\frac{1}{3}\, 9^{\frac{3}{2}}\right) - \left(\frac{1}{3}\, 1^{\frac{3}{2}}\right)$$

$$\frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

$\sqrt[2]{9^3}$

① sqr root first

② cube/exponent

Example: Find $\int_0^1 \frac{x-3}{x^2-6x+7}\ dx$ giving answer in the form $a \ln p$ where $a, p \in \mathbb{R}$.

x's & v's must correspond.

$v = x^2 - 6x + 7$

$\frac{dv}{dx} = 2x - 6$

$dx = \frac{dv}{2x-6}$

$x = 1 = 1 - 6 + 7 = 2$

$x = 0 = v = 7$

$$\int_7^{2} \frac{x-3}{v} \frac{dv}{2x-6}$$

$\frac{x-3}{2x-6} \rightarrow \frac{x-3}{2(x-3)}$

$$\frac{1}{2}\int_7^2 \frac{1}{v}\, dv$$

brackets where you integrate, yk?

$$= \frac{1}{2}\left[\ln v\right]_7^2 =$$

$$= \frac{1}{2}(\ln 2 - \ln 7)$$

$$= \frac{1}{2}\ln\frac{2}{7}$$

One more integration strategy before we move on...

Numerator is derivative of denominator

If an integral is of the form: $\int \frac{f'(x)}{f(x)} \quad dx$...

... then we can just write down the antiderivative as $F(x) = \ln(f(x)) + C$.

Words: If numerator is the derivative of the denominator, then antiderivative/integral is ln(denominator) +C

Example: $\int \frac{2x-3}{x^2-3x+4} \quad dx$

$v = x^2-3x+4$

$\frac{dv}{dx} = 2x-3$

$dx = \frac{dv}{2x-3}$

$$\int \frac{\cancel{2x-3}}{v} \frac{dv}{\cancel{2x-3}} = \int \frac{1}{v}\, dv = \ln v + C$$

$$= \ln(x^2-3x+4) + C$$

Example: $\int \frac{3x^2e^{x^3}}{e^{x^3}} \quad dx$

$y = e^{x^3}$ $y' = 3x^2e^{x^3}$

$$\cancel{\ln e}^{x^3} + C = x^3 + C$$

(cancel)

Example: $\int \frac{x-2}{x^2-4x+8} \quad dx$

$v = x^2-4x+8$

$\frac{dv}{dx} = 2x-4$

$dx = \frac{dv}{2x-4}$

$$\int \frac{\cancel{x-2}}{v} \frac{dv}{\cancel{2x-4}} = \frac{1}{2}\int \frac{1}{v}\, dv = \frac{1}{2} \ln x^2-4x+8 + C$$

since you need to factor - just use u-sub.